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Compatible triangulations and point partitions by series-triangular graphs

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Abstract

We introduce series-triangular graph embeddings and show how to partition point sets with them. This result is then used to prove an upper bound on the number of Steiner points needed to obtain compatible triangulations of point sets. The problem is generalized to finding compatible triangulations for more than two point sets and we show that such triangulations can be constructed with only a linear number of Steiner points added to each point set. Moreover, the compatible triangulations constructed by these methods are regular triangulations.

Keywords: Compatible triangulations; Steiner points; Series-triangular graphs

1. Introduction

Given two n point sets in general position in the plane, it is an open question as to whether compatible triangulations always exist between them. This problem was first posed by Aichholzer et al. [1] in 2002 and in a slightly different version by Saalfeld [6] in 1987. Aronov, Seidel, and Souvaine [2] studied the related problem of compatibly triangulating simple polygons and showed that $\Omega(n^2)$ Steiner points were necessary in some cases. An immediate corollary of Euler's Theorem states that the number of triangles in any triangulation of a set S with n points is $2n - 2 - h$, where h is the number of extreme points. Trivially, two triangulations must have the same number of triangles to be compatible so it is a necessary condition for a compatible triangulation that the point sets have the same number of extreme points. Aichholzer et al. conjecture that this condition is also sufficient. They also show that compatible triangulations, for sets obeying these necessary conditions, are always possible if one is allowed to add extra points, called *Steiner* points, to the sets. The number of Steiner points required by Aichholzer et al. is equal to the number of interior points of the set minus three.

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In this paper, a new method is proposed for obtaining compatible triangulations using Steiner points by allowing the addition of points both inside and outside the convex hull. This method requires the use of a number of Steiner points approximately equal to half the number of points in the set. We require that the points placed outside the convex hull be *close* to the convex hull. This variant on the problem is quite natural, extending the domain of problem instances to include sets for which the number of extreme points in the two sets are not equal. The method also allows for control over the structure of the Steiner point triangulation. In particular, the Steiner points and the edges between them form a weighted Delaunay triangulation, also known as a *regular* triangulation.

The problem of finding d-way compatible triangulations was considered by Krasser [5]. This natural generalization of the problem asks how d distinct point sets can be triangulated so that the resulting triangulations are pairwise compatible. In [5], it is shown that for $d > 2$, there exist d sets of n points which do not admit d-way compatible triangulations. It follows that the addition of Steiner points is *necessary* in order to yield a compatible triangulation. We extend our Steiner point method for the $d = 2$ case to produce a technique for d-way compatible triangulation using a number of added points to each set which is independent of d.

2. Steiner points

2.1. A *triangulation* of a set S of points in the plane $\mathbb{R}^2$, denoted by τ_S, is a maximal set of line segments between the points such that any pair of segments intersect at a common endpoint or not at all. When we refer to the triangles in a triangulation, we mean the *empty* triangles, those that do not contain any other points of S. This paper will only consider point sets S in *general position*, where no three points of S are collinear.

Let $CH(S)$ denote the vertices of the polygonal boundary of the convex hull of S under a cyclic ordering. The *orientation* of $CH(S)$ refers to whether this ordering is clockwise or counterclockwise around the polygon. The set of points in $CH(S)$ are called the *extreme points*, and the remaining points of S are called *interior points*.

Definition 1. Given two point sets S and T with $|S| = |T|$, along with triangulations τ_S and τ_T, we say τ_S and τ_T are *compatible* if there exists a bijection $f : S \to T$ such that $\triangle(a, b, c)$ is a clockwise-oriented triangle of τ_S if and only if $\triangle(f(a), f(b), f(c))$ is a clockwise-oriented triangle of τ_T.

Fig. 1 gives an example of compatible triangulations. There are variants of this definition which omit the requirement regarding orientation, requiring only that the bijection maps empty triangles to empty triangles. We have chosen the above definition because it is more useful for applications in computer graphics and cartography. It is important to note that none of these definitions require only the underlying graph structures of the triangulations to be isomorphic. However, we give a definition equivalent to Definition 1 in terms of the graph structure of the triangulations.

Definition 2. Given two point sets S and T along with triangulations τ_S and τ_T, we say τ_S and τ_T are *compatible* if the there exists an isomorphism between their underlying graphs that also maps $CH(S)$ to $CH(T)$ preserving their orientation in the plane.

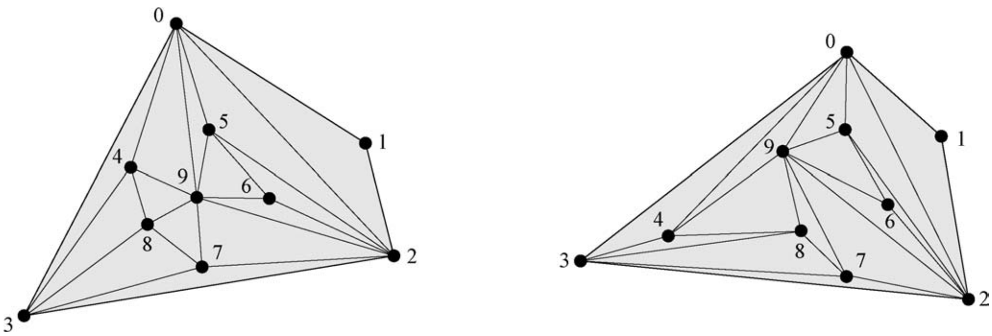

Fig. 1. Example of compatible triangulations.

2.2. The problem of finding compatible triangulations can be made easier if we can add extra points to the point sets, called *Steiner* points. Steiner points placed inside (outside) the convex hull of a point set are named *interior* (*exterior*) Steiner points. Aichholzer et al. [1] give a method of compatibly triangulating two point sets with the introduction of a number of interior Steiner points equal to the number of interior points minus three. One might try methods of compatible triangulation using exterior Steiner points, that is, additional points added outside of the convex hull of the original point set. Motivated by [3], we start by showing a method of doing this that uses a number of Steiner points independent of the size of the point sets to be triangulated.

Theorem 3. *Given two point sets S and T with $|S| = |T| = n$, then S and T may be compatibly triangulated with the addition of two Steiner points to each set.*

Proof. Order each set in terms of increasing second coordinate, and when points have the same second coordinate, order them in terms of decreasing first coordinate. Explicitly,

$$S = \{p_1 = (x_1, y_1), \dots, p_n = (x_n, y_n)\},$$

where if $i > j$, then either $y_i > y_j$ or $y_i = y_j$ and $x_i < x_j$. Similarly,

$$T = \{q_1 = (u_1, v_1), \dots, q_n = (u_n, v_n)\},$$

where if $i > j$, then either $v_i > v_j$ or $v_i = v_j$ and $u_i < u_j$. Add a Steiner point p_L slightly below p_1 and far enough to the left of S so that edges $p_i p_{i+1}$ (between consecutive points of S) and $p_L p_i$ (between p_L and points of S) do not intersect pairwise. Similarly, add a Steiner point p_R slightly above p_n and to the right of S with the corresponding property; see Fig. 2. Note that p_L and p_R exist since placing them arbitrarily far away yields edges arbitrarily close to horizontal. Add q_L and q_R to T in the same manner.

Let $S_* = S \cup \{p_L, p_R\}$ and $T_* = T \cup \{q_L, q_R\}$. By construction, the edges $p_L p_i$ and $p_i p_R$ (connecting each Steiner point to every point of the original set) together with the edges $p_i p_{i+1}$ yield a triangulation of S_*; a similar construction gives a triangulation to T_*. The bijection $f(p_*) = q_*$ shows these triangulations to be compatible. □

The preceding theorem demonstrates the power of adding exterior Steiner points. The downside is that the new Steiner points can be arbitrarily far away; a better solution would bound the maximum distance between points. The *radius* of a point set S, denoted by $r(S)$, is the radius of the smallest disk that contains all of S (and thus the convex hull of S). We want to bound the impact of placing points outside the convex hull, and so we employ this definition of radius since it is an approximate measure of the size of the convex hull of a point set in $\mathbb{R}^2$. In Section 4, we present a method of compatibly triangulating point sets where the Steiner points increase the radius of the point set at most by a constant factor.

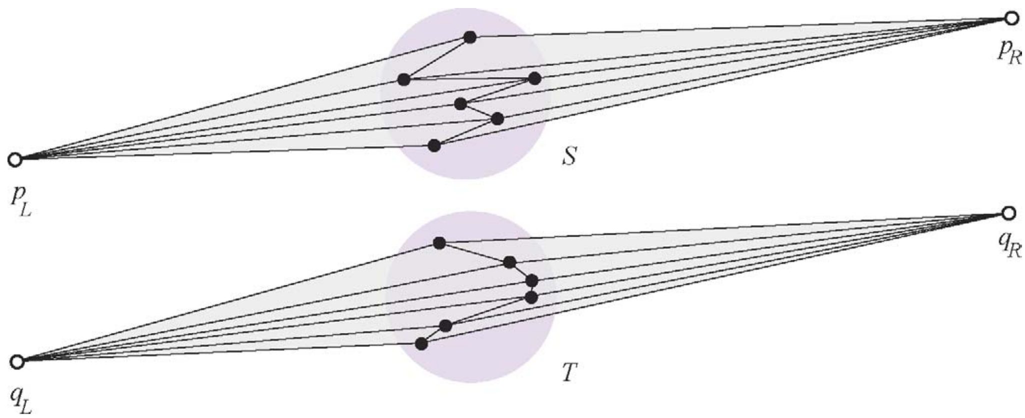

Fig. 2. Example of sufficiency of two exterior Steiner points.

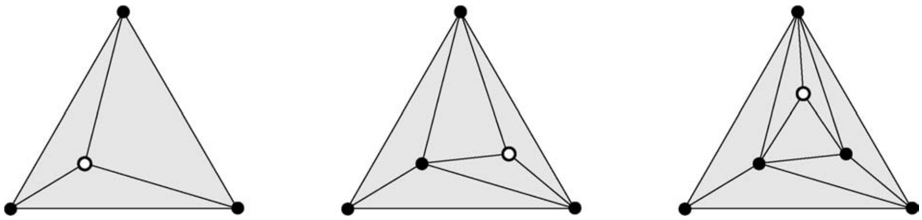

Fig. 3. Constructing series-triangular graphs one vertex at a time.

3. Partitioning point sets with series-triangular graph embeddings

3.1. In order to construct compatible Steiner point triangulations, it is helpful to break the problem into smaller pieces. One way is to lay down the Steiner points so that a triangulation of the Steiner points alone divides the original point sets into smaller point sets in each triangle. This only works if the triangulations on the Steiner points are compatible and the same number of points fall in corresponding triangles. In this section we show this is always possible.

Recall that Definition 2 states that compatibility is just an extreme point preserving isomorphism. Therefore, in order to add Steiner points that can be compatibly triangulated, we view these triangulations as two embeddings of the same graph in which the same vertices map to the extreme points in the same orientation. The graphs used to perform this desired partition all have a common structure.

Definition 4. A graph $G(V, E)$ is *series-triangular* if

(1) it is planar,
(2) every embedding of G in $\mathbb{R}^2$ is a triangulation, and
(3) when $|V| > 3$, there is a vertex v in V such that $G \setminus v$ is series-triangular.

Fig. 3 shows the construction of a series triangular graph, starting with a triangle, one vertex at a time. The recursive definition above implies that there is a labeling $v_1, \ldots, v_n$ of the vertices of G such that the subgraph G_i induced on $\{v_1, \ldots, v_i\}$ is series-triangular for $i \geqslant 3$; this is called an *ordered labeling*. In particular, if an ordered labeling is given, then it is possible to embed G in $\mathbb{R}^2$ so that $\{v_1, v_2, v_3\}$ map to extreme points oriented clockwise. An embedding that satisfies this property is called an *ordered embedding*. An immediate consequence of Definition 2 is the following useful lemma.

Lemma 5. *Let ϕ, ψ be straight-line embeddings of a series-triangular graph G. Then $\phi(G)$ and $\psi(G)$ are compatible triangulations under the bijection $f(\phi(v)) = \psi(v)$ if and only if ϕ and ψ map the same vertices of G to extreme points in the embedded triangulations with the same orientation.*

In other words, for a given ordered labeling of the vertices of G, all ordered embeddings are compatible. Thus, once an ordered labeling for G (there are many) is chosen, the compatibility class of the ordered embedding is fixed. In what follows, we always assume an ordered labeling is given with G and refer to the triangles in an ordered embedding of G as simply the triangles of G.

3.2. As series-triangular graph embeddings are used to construct compatible Steiner point triangulations, it is natural to desire also that the triangulations being constructed have a nice structure. The following gives such a result.

Lemma 6. *A straight-line embedding of a series-triangular graph is a regular triangulation.*

Proof. Recall that a triangulation τ of a point set S in $\mathbb{R}^2$ is *regular* if it is the projection of the lower convex hull of a polytope in $\mathbb{R}^3$. We construct such a polytope by inductively defining a height function $h: S \to \mathbb{R}$. Order the points of $S = \{x_1, \ldots, x_n\}$ such that τ is the ordered embedding. Let $S' = \{(x_i, h(x_i))\}$ be the vertices of the convex polytope P in $\mathbb{R}^3$, and let P_i be polytope at the ith step, the convex hull of the first i points in S'. As a base case, since a single

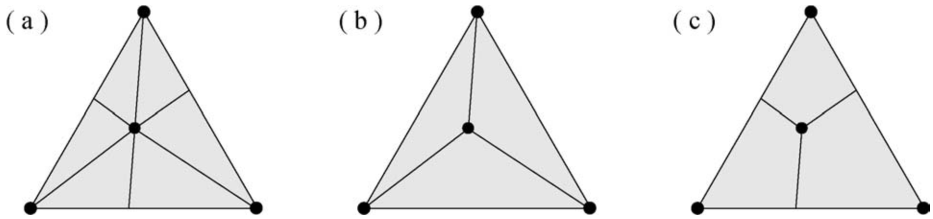

Fig. 4. (a) Subdivision, (b) trisection, and (c) Y-section of a triangle.

triangle is always regular, set $x_1 = x_2 = x_3 = 1$. For each x_i, let $\triangle(x_a, x_b, x_c)$ be the triangle that is trisected by x_i. Pick z so the point $(x_i, z) \in \mathbb{R}^3$ lies on the plane passing through $\triangle(x_a, x_b, x_c)$. Then we can choose ε small enough such that setting $h(x_i) = z - \varepsilon$ places $(x_i, h(x_i))$ close enough to the surface of P_i, where the only new triangles added to the convex hull are exactly those that trisect $\triangle(x_a, x_b, x_c)$. By induction, the entire height function can be constructed. □

3.3. Before analyzing issues with series-triangular graphs, we first examine the partition of point sets within a single triangle.

Definition 7. Let p be an interior point of a triangle T in $\mathbb{R}^2$. The three lines passing through p and each vertex of T *subdivides* T into six triangles. The *trisection* of T at p is obtained from the three line segments from each vertex of T to p. The three halflines emanating from p directed away from the three vertices of T give the *Y-section* of T at p, denoted as $Y(p)$; see Fig. 4.

If a Steiner point d is placed inside T, the trisection of T at d divides T into three adjacent triangles. Ideally, we would like to have the freedom to place d in such a way that the number of points in each of the triangles of the trisection is prescribed. The following shows this is always possible.

Lemma 8. *Let $S = \{p_1, \ldots, p_n\}$ be a set of points inside a triangle $T = \triangle(a, b, c)$ satisfying the following conditions*:

(1) *The points $S \cup \{a, b, c\}$ are in general position.*
(2) *If A is the arrangement of all lines passing through pairs of points (t, s) where $t \in \{a, b, c\}$ and $s \in S$ then the lines of A have no three-way intersection inside T.*

Then the set $Y(p_1) \cup \cdots \cup Y(p_n)$ divides T into $\binom{n+2}{2}$ regions.

Proof. We construct this inductively, starting with T and adding a point of S at each step. It is clear from the definition that every pair of distinct Y-sections intersect at exactly one point. It follows from the assumption that no three Y-sections intersect at a point. Thus, k distinct Y-sections result in $k - 1$ intersections. Each additional Y-section divides the region of its center point into three parts and divides each of the $k - 1$ other regions into two parts, one for each line crossed; Fig. 5 shows some examples. Thus $\sum(i + 1) = \binom{n+2}{2}$ is the total number of regions for n points. □

We also need the following lemma, which is a special case of a theorem by García and Tejel [4].

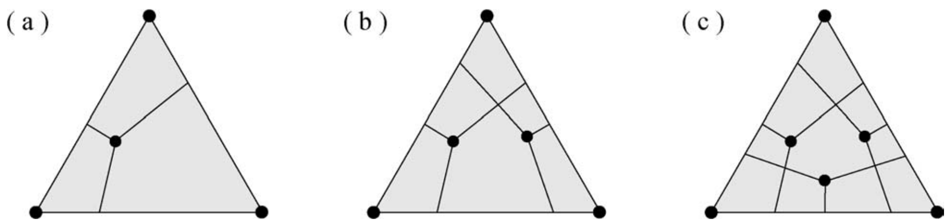

Fig. 5. The Y-sections of (a) one, (b) two and (c) three points.

Lemma 9. *Let $S = \{p_1, \ldots, p_n\}$ be a set of points inside a triangle T satisfying the conditions of Lemma* 8. *Let x, y, z be positive integers such that $x + y + z = n$. A Steiner point d can be placed in T such that the three triangles formed by the trisection of T at d contain x, y, z interior points respectively.*

3.4. We now show how a series-triangular graph can be embedded in the plane so that the triangular faces of the embedding partition a set of points nicely.

Theorem 10. *Let S be an n point set in $\mathbb{R}^2$ and $G(V, E)$ be a series-triangular graph, and let k be odd. If $\{t_1, \ldots, t_k\}$ are the triangles of G and $n_1 + \cdots + n_k$ is a partition of n, then there is an ordered embedding $\phi(G)$ with the property that exactly n_i points of S lie in the triangle t_i.*

Proof. We construct ϕ by induction on $|V| = \frac{k+5}{2}$. The base case when $|V| = 3$ and $k = 1$ is trivial because any triangle enclosing all of S suffices. We choose this triangle so that it satisfies the conditions of Lemma 8. Now, assume $k \geqslant 3$ and let $n' = n_{k-2} + n_{k-1} + n_k$. As G is series-triangular, let v be the vertex such that when removed along with all edges touching it, the resulting graph, called G', is series-triangular. Since G' has one less vertex and two less triangles, by induction we can find an embedding $\phi'(G')$ such that $n_1, n_2, \ldots, n_{k-3}, n'$ points of S lie in the respective triangles $t'_1, t'_2, \ldots, t'_{k-3}, t'_{k-2}$ of $\phi'(G')$. Further assume by induction that so far the conditions of Lemma 8 are met. Now, Lemma 9 ensures that there is a non-empty open region inside t'_{k-2}, any point of which, when connected to the vertices of t'_{k-2}, will divide the triangle into three smaller triangles having exactly n_{k-2}, n_{k-1} and n_k of the n' points of $t'_{k-2} \cap S$. From this region we may pick a point p so that when we define ϕ by extending ϕ' to include $\phi(v) = p$, the conditions of Lemma 8 are satisfied. The three new triangles formed by the addition of $\phi(v)$ are the triangles t_{k-2}, t_{k-1}, t_k of the embedding $\phi(G)$.

In a sense, this construction stretches the triangulation over the point set controlling how many points land in each triangle. It is not hard to see that this partitioning theorem can be modified so that the number of points in the unbounded face can also be specified.

4. Compatible Steiner point triangulations

4.1. We will now show that the partitioning theorem in the previous section can be used to triangulate point sets with Steiner points. This method will embed a series-triangular graph around each point set, partitioning them so each triangle receives one point. The new points introduced will be the Steiner points and the compatible triangulations of those points will be extended to compatible triangulations of the entire point set.

A partition of points by a series-triangular graph embedding encloses the entire point set in a triangle. The smallest triangle containing the disk of radius $r(S)$ has radius $2r(S)$, so the radius of the point set increases at most by a factor of 2 by adding the points needed to partition it.

Theorem 11. *Given point sets S, T with $|S| = |T| = n$, it is possible to compatibly triangulate S and T with the addition of at most $\frac{n}{2} + 3$ Steiner points such that the radii of the point sets increase at most by a factor of* 2.

Proof. Let G be any series-triangular graph with at least n bounded triangles. Use Theorem 10 to obtain an ordered embedding of G in $\mathbb{R}^2$ so that no two points of S share a triangle; repeat this process for T. By Lemma 5, the embeddings are compatible triangulations. Choose the partitions so that compatible triangles contain the same number of vertices (0 or 1 in this case). Because each pair of compatible triangles admit compatible triangulations of the interior points, these triangulations are easily extended to compatible triangulations of the entire point sets. Since any planar triangulation on k vertices has $2k - 5$ bounded triangles, we can choose G such that $|V| = \lceil \frac{n+5}{2} \rceil < \frac{n}{2} + 3$. $\square$

Remark. In light of Lemma 6, the triangulations constructed above are regular.

4.2. Aichholzer et al. [1] show that for point sets with only three interior points, there exists compatible triangulations as long as the point sets have the same number of extreme points. Moreover, this result holds even if the bijection between their extreme points is prescribed by cyclic rotation. An immediate consequence of this fact is that the proof of the preceding theorem can be modified to place three points in each triangle rather than one. Thus we get the following:

Corollary 12. Given point sets S, T with $|S| = |T| = n$, it is possible to compatibly triangulate S and T with the addition of at most $\frac{n}{6} + 3$ Steiner points such that the radii of the point sets increase at most by a factor of 2.

Computer simulations in [1] have shown that all sets of 9 points with the same number of extreme points yield compatible triangulations even if the bijection between the extreme points is fixed by cyclic rotation. Thus, this result can again be refined by placing 6 points in each triangle.

Corollary 13. Given point sets S, T with $|S| = |T| = n$, it is possible to compatibly triangulate S and T with the addition of at most $\frac{n}{12} + 3$ Steiner points such that the radii of the point sets increase at most by a factor of 2.

These corollaries demonstrate the expandability of the series-triangular partition method. As larger point sets are shown to yield compatible triangulations with extreme points prescribed by cyclic rotation, the number of Steiner points needed to compatibly triangulate goes down. Our methods give a framework for directly extending results on small point sets to arbitrary point sets.

5. d-way compatible triangulations

5.1. We apply our methods to a more general question posed by Krasser in [5], where there are not just two but d point sets $S_1, \ldots, S_d$. Thus, instead of a single bijection, a set of bijections $f_1, \ldots, f_{d-1}$ are needed with $f_i : S_i \to S_{i+1}$ mapping triangulations to triangulations compatibly. An important application of compatible triangulations comes from the problem of *morphing* computer graphics. Surazzhsky and Gotsman [7] show that if a pair of compatible triangulations are given, then it is possible to linearly morph one into the other without any triangle edges intersecting. Suppose one desires to morph a triangulated point set S_1 to a point set S_2 and then again to third point set S_3 and so on to S_d. In such a case, d-way compatible triangulation are necessary. Indeed, previous methods for finding compatible Steiner triangulations would require adding new Steiner points for each morph thus making the number of Steiner points dependent on d. However, it is shown below that our method for compatible Steiner triangulation of two point sets extends to d-way compatible triangulations so that the number of Steiner points added to each set stays linear and is independent of d.

Definition 14. Given point sets $S_1, \ldots, S_d$, with $|S_i| = |S_j|$ for all i, j, and triangulations $\tau_1, \ldots, \tau_d$, then $\{\tau_1, \ldots, \tau_d\}$ is *d-way compatible* if τ_i and τ_j are compatible for all i, j.

As stated earlier, it is not known whether Steiner points are necessary for a compatible triangulation of two point sets. The open conjecture is that the compatible triangulations always exist when the point sets are the same size and have the same number of extreme points. The corresponding conjecture for d-way compatible triangulation is false even for the case when $d = 3$; see [5] for a counterexample.

5.2. Even in simple cases, Steiner points are necessary for d-way compatible triangulations. We show the number of Steiner points needed for each point set to be independent of d.

Theorem 15. *Given point sets $S_1, \ldots, S_d$ with $|S_i| = n$, it is possible to d-way compatibly triangulate $S_1, \ldots, S_d$ with the addition of at most $\frac{n}{2} + 3$ Steiner points to each set such that the radii of the point sets increase by at most a factor of* 2.

Proof. Construct any series-triangular graph $G(V, E)$ with at least n triangles. By Theorem 10, it is possible to embed G in $\mathbb{R}^2$ for each S_i so that one point of S_i falls in each triangle. There may be one empty triangle if n is even but this is inconsequential as long as we choose the same triangle to remain empty in every embedding. All d embeddings $\phi_1, \ldots, \phi_d$ are compatible by Lemma 5.

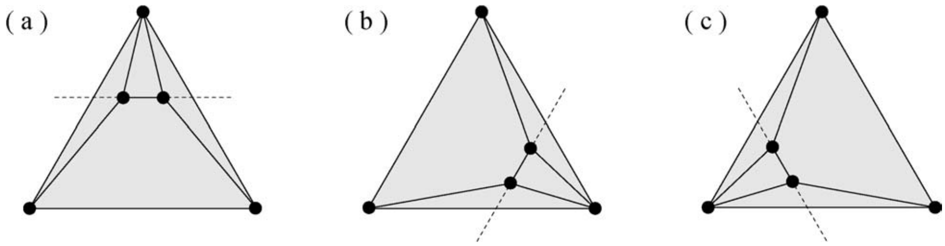

Fig. 6. Three point sets in Steiner point triangles and their forced edges.

Let $\{t_1, \ldots, t_n\}$ be the triangles of G. For each $p \in S_i$ and $q \in S_{i+1}$, let $f_i(p) = q$ if and only if p is inside triangle $\phi_i(t_j)$ and q in $\phi_{i+1}(t_j)$ for some j. Extend the bijections to include the Steiner points by letting $f_i(\phi_i(v)) = \phi_{i+1}(v)$. All remaining edges are forced and compatible; they are simply the edges connecting each point in the original sets to the three vertices of the triangle that encloses it. As in the proof of Theorem 11, one can construct G so that $|V| \leqslant \frac{n}{2} + 3$. The Steiner points for each point set S_i are $\phi_i(V)$; thus the number of Steiner points required for each set is also at most $\frac{n}{2} + 3$. $\square$

Unlike the case where $d = 2$, it is not possible to extend this method by placing more points inside each triangle of the embedded graph. We note that if even two points go into any one triangle, it may be impossible to the extend the compatible triangulation of the Steiner points to a compatible triangulation of the entire point sets. Fig. 6 gives a simple example where this is the case. Note that at least one of the extreme points must have degree three; this is not possible however since each of the extreme points has four forced edges in at least one embedding.

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